

# Gurevich-Zybin system

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ABSTRACT. We present three different linearizable extensions of the Gurevich-Zybin system. Their general solutions are found by reciprocal transformations.

In this paper we rewrite the Gurevich-Zybin system as a Monge-Ampere equation. By application of reciprocal transformation this equation is linearized. Infinitely many local Hamiltonian structures, local Lagrangian representations, local conservation laws and local commuting flows are found. Moreover, all commuting flows can be written as Monge-Ampere equations similar to the Gurevich-Zybin system.

The Gurevich-Zybin system describes the formation of a large scale structures in the Universe. The second harmonic wave generation is known in nonlinear optics. In this paper we prove that the Gurevich-Zybin system is equivalent to a degenerate case of the second harmonic generation. Thus, the Gurevich-Zybin system is recognized as a degenerate first negative flow of two-component Harry Dym hierarchy up to two Miura type transformations. A reciprocal transformation between the Gurevich-Zybin system and degenerate case of the second harmonic generation system is found. A new solution for the second harmonic generation is presented in implicit form.

to the memory of Professor Andrea Donato (Messina University)

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# 1. Introduction

The invisible nondissipative dark matter plays a decisive role in the formation of a large scale structures in the Universe: galaxies, clusters of galaxies, superclusters. Corresponding nonlinear dynamics can be described (see [14]) by following hydrodynamic-like system

$$\rho_t + \nabla(\rho \mathbf{u}) = 0, \quad u_t + (\mathbf{u} \nabla)u + \nabla \Phi = 0, \quad \Delta \Phi = \rho, \quad (1)$$

where the first two equations are usual hydrodynamic equations (the continuity equation and the Euler equation, respectively), but the third equation is the famous Poisson equation. At least for the first time this system was derived by J.H. Jeans (see [16] and also [31]) for a description of instabilities of a homogeneous distribution of a matter.

Such dynamics of dissipationless gravitating gas is a special limit ( $\varepsilon \rightarrow 0$ ) of another system (another sign of  $\rho$  is unessential)

$$\rho_t + \nabla(\rho \mathbf{u}) = 0, \quad u_t + (\mathbf{u} \nabla)u + \nabla \Phi = 0, \quad \Delta \Phi = \varepsilon e^\Phi - \rho$$

describing fully nonlinear flows in a two temperature unmagnetized collisionless plasma in dimensionless variables (nonlinear ion-acoustic waves; see for instance [30]).

The main advantage of the Jeans theory is a reckoning of two factors: a gravity attracting a matter in separate lumps and clots; a pressure decreasing an inhomogeneity of a matter in the Universe.

Recently a new achievement in the investigation of the system (1) was made (see [14]) in Cosmology. The nonlinear one-dimensional dynamics of a dark matter is described by the equations [14]

$$u_t + uu_x + v = 0, \quad v_t + uv_x = 0, \quad (2)$$

where  $\rho = v_x$ ,  $v = \Phi_x$ . The analysis of equations (2) in a multimode form demonstrates the transition from the hydrodynamic to the equilibrium kinetic state [14]. It means, that the exact solution of the equations (2) describes a fundamental physical process (see [14] again).

It is amazing that the *inhomogeneous* hydrodynamic type system (2) can be integrated, up to the first singularity, by the Hodograph Method (see [14]). For this reason further we will call the system (2) the Gurevich-Zybin system emphasizing that the one-dimensional reduction (2) of the system (1) is integrable.

Here we give the general solution by the method of *Reciprocal Transformations*. Moreover, we present three different linearizable extensions of this system (2) with their general solutions given by corresponding reciprocal transformations. Actually these reciprocal transformations have clear pure mathematical (hodograph method) and physical (transition from Euler to Lagrange variables) interpretations. In the next section we present three linearizable extensions of the Gurevich-Zybin systems with their general solutions. In second section relationship between two-component generalization of the Hunter-Saxton equation and the Gurevich-Zybin system is established. In the third section the Gurevich-Zybin system is rewritten as a Monge-Ampere equation (following the approach developed by Andrea Donato). In the fourth section a bi-Hamiltonian structure of the Gurevich-Zybin system is found (following the approach developed by Yavuz Nutku). In the fifth section, by the application of a reciprocal transformation, a simplest recursion operator is constructed. Infinitely many local conservation laws, local commuting flows, local Lagrangians and local Hamiltonians are found. Moreover, all commuting flows are Monge-Ampere equations. Thus, the Gurevich-Zybin system is a member of an

integrable hierarchy of Monge-Ampere equations. In sixth section a bi-Hamiltonian formulation for the Gurevich-Zybin system is given in a canonical form. The Gurevich-Zybin system is recognized as a first negative flow of two-component Harry Dym hierarchy. In seventh section Miura type and reciprocal transformations between the Gurevich-Zybin system and Kaup-Boussinesq hierarchy are given. In eighth section we finally prove that the Gurevich-Zybin system is equivalent to a degenerate case of the second harmonic generation system up to above-mentioned transformations. A new solutions of the second harmonic generation system is found. In conclusion we discuss about sort of integrable problems belonging some different hierarchies of integrable equations.

## 2. General Solution

The Gurevich-Zybin system (1) in one dimensional case precisely has a form

$$\rho_t + \partial_x(\rho u) = 0, \quad u_t + uu_x + \Phi_x = 0, \quad \Phi_{xx} = \rho. \quad (3)$$

This system can be generalized *at least* in three different forms:

$$\rho_t + \partial_x(\rho u) = 0, \quad u_t + uu_x + \mu'''(\Phi_x) = 0, \quad \Phi_{xx} = \rho, \quad (4)$$

$$\rho_t + \partial_x(\rho u) = 0, \quad u_t + uu_x + \Phi_x = 0, \quad \Phi_{xx} = B(\rho), \quad (5)$$

$$\rho_t + \partial_x(\rho u) = 0, \quad u_t + uu_x + \mu'''(\Phi_x) = 0, \quad \partial_x C(\Phi_x) = \rho, \quad (6)$$

where  $\mu(z)$ ,  $B(\rho)$  and  $C(z)$  are arbitrary functions. It is interesting that is not obvious that the system (4) is equivalent to the Gurevich-Zybin system (3). Indeed, the system (4) written like (2)

$$z_t + uz_x = 0, \quad u_t + uu_x + \mu'''(z) = 0 \quad (7)$$

is exactly (2) up to point transformation  $v = \mu'''(z)$ . One can introduce reciprocal transformation

$$dz = \rho dx - \rho u dt, \quad d\tau = dt. \quad (8)$$

Then  $\partial_x = \rho \partial_z$  and  $\partial_t = \partial_\tau - \rho u \partial_z$ . Thus, the system (4) has the form

$$\left(\frac{1}{\rho}\right)_\tau = u_z, \quad u_\tau = -\mu'''(z), \quad (9)$$

in new variables, where  $z = \Phi_x$ . Thus, the general solution of the system (9) is

$$u = -\mu'''(z)\tau + D'(z), \quad \frac{1}{\rho} = -\mu''''(z)\tau^2/2 + D''(z)\tau + E''(z),$$

where  $D(z)$  and  $E(z)$  are arbitrary functions. Finally the general solution of the system (4) can be given in implicit form

$$\begin{aligned} u &= -\mu'''(z)t + D'(z), & \rho &= [-\mu''''(z)t^2/2 + D''(z)t + E''(z)]^{-1}, \\ x &= -\mu'''(z)t^2/2 + D'(z)t + E'(z), \\ \Phi &= (\mu''(z) - z\mu'''(z))t^2/2 + (zD'(z) - D(z))t + zE'(z) - E(z), \end{aligned} \quad (10)$$

where  $z$  is a parameter here.

Above reciprocal transformation applied to the system (5) yields

$$\left(\frac{1}{\rho}\right)_\tau = u_z, \quad u_\tau = -v, \quad v_z = B(\rho)/\rho,$$

where  $v = \Phi_x$ . If function  $\rho$  can be explicitly expressed from algebraic equation

$$\tau = G(z) - \int^{\rho} \frac{d\theta}{\theta^2 \sqrt{2 \int B(\theta) \theta^{-3} d\theta - F(z)}},$$

where  $F(z)$  and  $G(z)$  are arbitrary functions, then the general solution of the system (5) can be obtained. For instance, if  $B(\rho) = \rho$ , then a general solution is already given by (10) (remember that in such sub-case systems (4) and (5) coincide if  $\mu(z) = z^4/24$ ); in simplest perturbed case  $B(\rho) = \rho + \delta/\rho$  ( $\delta = \text{const}$ ) the general solution is expressed via Weiershtrass elliptic functions

$$\rho = \frac{\delta}{6} [\wp(\tau - G(z), \frac{\delta}{3}, \frac{\delta^2}{36} F(z))]^{-1}, \quad u_z = \frac{6}{\delta} \wp', \quad v_z = 1 - \frac{36}{\delta} \wp^2.$$

The reciprocal transformation (8) applied to the system (6) yields

$$\left(\frac{1}{\rho}\right)_{\tau} = u_z, \quad u_{\tau} = -\mu'''(v), \quad C(v) = z,$$

where  $v = \Phi_x$ . Thus, the general solution of this system is

$$\begin{aligned} v &= V(z), & u &= D'(z) - \mu'''(v)\tau, \\ \rho &= [E''(z) + D''(z)\tau - \frac{\mu''''(v)}{2C'(v)}\tau^2]^{-1}, \\ x &= E'(z) + D'(z)\tau - \mu'''(v)\tau^2/2, \end{aligned}$$

where  $D(z)$ ,  $E(z)$  are arbitrary functions and  $V(z)$  is inverse function to  $C(v)$ . Finally the general solution of the system (6) can be given in implicit form

$$\begin{aligned} v &= V(z), & u &= D'(z) - \mu'''(v)t, \\ \rho &= [E''(z) + D''(z)t - \frac{\mu''''(v)}{2C'(v)}t^2]^{-1}, \\ x &= E'(z) + D'(z)t - \mu'''(v)t^2/2, \\ \Phi &= zE'(z) - E(z) + (zD'(z) - D(z))t + [G(v) - zG'(v)]t^2/2, \end{aligned}$$

where  $G' = \mu'''C'$  and  $z$  is a parameter here. Also the system (6) can be written in hydrodynamic like form

$$\rho_t + \partial_x(\rho u) = 0, \quad u_t + uu_x + \frac{1}{\rho}\partial_x P = 0, \quad (11)$$

where the pressure  $P$  is a nonlocal function of the density  $\rho$

$$P = P(V(\partial_x^{-1}\rho)).$$

In particular case  $C(v) = v$  the system (6) coincides with the system (4); the system (6) was written in form (11) in [6] for particular case  $\mu(v) = v^4/24$  and  $C(v) = v$ .

### 3. Two-component generalization of the Calogero equation

The simplest two-component linearizable generalization

$$\eta_t + \partial_x(\eta u) = 0, \quad u_{xt} + uu_{xx} + \Psi(\eta, u_x) = 0 \quad (12)$$

of the Calogero equation (see [7])

$$u_{xt} = uu_{xx} + R(u_x)$$

was presented (functions  $\Psi(a, b)$  and  $R(c)$  are arbitrary here) in [26]. Some particular cases of the Calogero equation like the Hunter-Saxton equation (i.e.  $R(c) = \nu c^2$ , where  $\nu = \text{const}$ ; see [15]) are interested from physical point of view. The Calogero equation is linearizable by a reciprocal transformation (see [26]). For instance, the Hunter-Saxton equation is related to the famous Liouville equation by a reciprocal transformation (see [8]). Thus, the system (12) is a natural generalization of the Liouville equation on two-component case up to module of a (invertible) reciprocal transformation.

The reciprocal transformation

$$d\zeta = \eta dx - \eta u dt, \quad dy = dt \quad (13)$$

applied to the system (12) yields the *ordinary* differential equation

$$s_{yy} + \Psi(e^{-s}, s_y) = 0,$$

where  $s = -\ln \eta$ , which can be reduced to the first order equation

$$ada + \Psi(e^{-s}, a)ds = 0, \quad (14)$$

where  $a = s_y = u_x$ . Then a general solution can be constructed in two steps from

$$u_\zeta = \left(\frac{1}{\eta}\right)_y, \quad dx = \frac{1}{\eta} d\zeta + u dy.$$

A solution  $q(s, a)$  of the linear equation

$$\frac{\partial q}{\partial s} = \frac{\Psi(e^{-s}, a)}{a} \frac{\partial q}{\partial a} \quad (15)$$

determined by the characteristic equation (14) yields extra conservation law

$$\rho_t + \partial_x(\rho u) = 0,$$

where  $\rho(\eta, u_x) = \eta \exp q$ . The comparison of the second equation in (12) and the second equation in (3) yields another relationship

$$\Psi = \rho + u_x^2.$$

Thus, a solution of the *nonlinear* equation (substitute  $\Psi$  from above equation into (15))

$$q_s = \left(a + \frac{e^{q-s}}{a}\right) q_a$$

describes a transformation between the Gurevich-Zybin system (3) and two-component generalization of the Hunter-Saxton equation (12).

**Remark 1:** Above equation under the substitution

$$q = s + \ln\left(\frac{n}{2}e^{-3s} - a^2\right)$$

transforms into well-known inhomogeneous Riemann-Monge-Hopf equation

$$n_y + nn_c = -\frac{2c}{9y^2},$$

where  $y = e^{-3s}/3$  and  $c = a^2$ . Its general solution can be given just in parametric form

$$n = \frac{1}{3}A_1(\xi)y^{-2/3} + \frac{2}{3}A_2(\xi)y^{-1/3}, \quad c = A_1(\xi)y^{1/3} + A_2(\xi)y^{2/3},$$

where  $\xi$  is a parameter and  $A_1(\xi)$ ,  $A_2(\xi)$  are arbitrary functions. However, the general solution of the equation of the first order depends by **one** function of a single variable only. It means, that if for instance  $A_1(\xi) \neq \text{const}$ , then by re-scaling  $A_1(\xi) \rightarrow \xi$  the general solution take a form

$$n = \frac{1}{3}\xi y^{-2/3} + \frac{2}{3}A(\xi)y^{-1/3}, \quad c = \xi y^{1/3} + A(\xi)y^{2/3},$$

where  $A(\xi)$  is an arbitrary function.

In a particular case the substitutions (see [26])

$$\rho = -\frac{1}{2}(u_x^2 + \eta^2), \quad \Psi(\eta, u_x) = \frac{1}{2}(u_x^2 - \eta^2) \quad (16)$$

connect the Gurevich-Zybin system (3) with two-component Hunter-Saxton system (see (12) and [25]), which is a *bi-directional* version of the Hunter-Saxton equation. A general solution of the Gurevich-Zybin system in field variables  $\eta$  and  $u$  has the implicit form with respect to the parameter  $\zeta$

$$\begin{aligned} \eta &= \left[ \frac{1}{k'(\zeta)} + \frac{1}{4}k'(\zeta)(t - m(\zeta))^2 \right]^{-1}, \\ u &= \frac{1}{2}tk(\zeta) - \frac{1}{2} \int m(\zeta)dk(\zeta), \\ x &= \int \left[ \frac{1}{k'(\zeta)} + \frac{1}{4}k'(\zeta)m^2(\zeta) \right]d\zeta + \frac{1}{4}t^2k(\zeta) - \frac{1}{2}t \int m(\zeta)dk(\zeta), \end{aligned} \quad (17)$$

where  $m(\zeta)$ ,  $k(\zeta)$  are arbitrary functions. One can substitute above expression for  $\eta$  to the first equation in (16) and compare expressions for  $(\rho, x, u)$  from (10) (in this case  $\mu'''(z) = z$ ) and above, then a relationship between the arbitrary functions  $k$ ,  $m$  and  $D$ ,  $E$  will be reconstructed.

#### 4. Reformulation of the Gurevich-Zybin system as a Monge-Ampere equation

A lot of physically motivated nonlinear systems can be written as Monge-Ampere equations (see [29]). To this moment we have no unique method for constructing such relationships. One simple approach was suggested to author by Andrea Donato (see [9]) at the ‘‘Lie Group Analysis’’ conference in Johannesburg (South Africa) at 1996.

The Gurevich-Zybin system in physical field variables (3) has four local conservation laws

$$\begin{aligned} u_t + \partial_x(u^2/2 + \Phi) &= 0, & \rho_t + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \Phi_x^2/2) &= 0, & \partial_t(\rho u^2 - \Phi_x^2) + \partial_x(\rho u^3) &= 0 \end{aligned}$$

as a consequence of obvious local Hamiltonian structure

$$v_t = \frac{\delta H_2}{\delta u}, \quad u_t = -\frac{\delta H_2}{\delta v}, \quad (18)$$

where the Hamiltonian is  $H_2 = \frac{1}{2} \int [-u^2 v_x + v^2] dx$ , the momentum is  $H_1 = \int u v_x dx$ , and two Casimirs are functionals  $Q_1 = \int u dx$  and  $Q_2 = \int \rho dx$  of corresponding Poisson bracket

$$\{\rho(x), u(x')\} = \{u(x), \rho(x')\} = \delta'(x - x'). \quad (19)$$

The existence of above first three local conservation laws is obvious. However, the fourth conservation law is not easy to check. Since  $\rho = \Phi_{xx}$ , then  $\rho u = -\Phi_{xt}$ , then  $\rho u^2 + \Phi_x^2/2 = \Phi_{tt}$ , then above fourth conservation law is valid. Eliminating physical field variables  $\rho$  and  $u$  from these three equations, the Monge-Ampere equation is given by

$$\Phi_{xx} \Phi_{tt} - \Phi_{xt}^2 = \frac{1}{2} \Phi_x^2 \Phi_{xx}. \quad (20)$$

In paper [14] the Gurevich-Zybin system was linearized by a hodograph method. A general solution has been presented too. Thus, this Monge-Ampere equation is linearizable and has the general solution in implicit form (see the end of section 2)

$$\Phi = -\frac{1}{4} v^2 t^2 + (v D'(v) - D(v)) t + v E'(v) - E(v), \quad x = -v t^2/2 + D'(v) t + E'(v),$$

where  $D(v)$  and  $E(v)$  are arbitrary functions,  $v$  is a parameter here.

Since the Gurevich-Zybin system can be written in the form (4), (7), we shall use an arbitrary value of function  $\mu(z)$  in next **two** sections.

## 5. Bi-Hamiltonian structure

The Gurevich-Zybin system (7) has local bi-Hamiltonian structure, where the first local Hamiltonian structure is (18)

$$z_t = \frac{\delta H_2}{\delta u}, \quad u_t = -\frac{\delta H_2}{\delta z}, \quad (21)$$

just three conservation laws are

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x[\rho u^2 + \mu''(\Phi_x)] &= 0, \\ \partial_t[\rho u^2 - 2\mu''(\Phi_x)] + \partial_x(\rho u^3) &= 0, \end{aligned}$$

associated with the first Poisson bracket (19). The Hamiltonian is  $H_2 = \int [-\rho u^2 + 2\mu''(\Phi_x)] dx$ , the momentum is  $H_1 = \int \rho u dx$  and the Casimir is  $H_0 = \int \rho dx$ . Corresponding Lagrangian representation is

$$S_1 = \int \left[ \frac{z_t^2}{2z_x} + \mu''(z) \right] dx dt, \quad (22)$$

where  $u = -z_t/z_x$ . Thus, the Lagrangian

$$S_1 = \int \left[ \frac{\Phi_{xt}^2}{2\Phi_{xx}} + \mu''(\Phi_x) \right] dx dt$$

creates the Euler-Lagrange equation

$$\Phi_{xx} \Phi_{tt} - \Phi_{xt}^2 = \mu''(\Phi_x) \Phi_{xx}, \quad (23)$$

which is a Monge-Ampere equation (cf. (20)).

At the same time (23) allows other Lagrangian representation (see [24])

$$S_2 = \int [\frac{1}{2}\Phi_{xx}\Phi_t^2 - \mu(\Phi_x)]dxdt. \quad (24)$$

Thus, the Monge-Ampere equation (23) has the second Hamiltonian structure

$$r_t = \partial_x \frac{\delta \bar{H}_2}{\delta z}, \quad z_t = \partial_x \frac{\delta \bar{H}_2}{\delta r}$$

determined by the local Poisson bracket

$$\{z(x), r(x')\}_2 = \{r(x), z(x')\}_2 = \delta'(x - x'), \quad (25)$$

where  $r = \Phi_{xx}\Phi_t$ , the Hamiltonian is  $\bar{H}_2 = \int [\frac{r^2}{2z_x} + \mu(z)]dx$ , the momentum is  $\bar{H}_1 = \int rzdx$ , two Casimirs are  $\bar{Q}_1 = \int rdx$  and  $\bar{Q}_2 = \int zdx$ . Four local conservation laws associate with above second Hamiltonian structure are

$$\begin{aligned} z_t &= \partial_x \left( \frac{r}{z_x} \right), & \partial_t \left[ \frac{r^2}{2z_x} + \mu(z) \right] &= \partial_x \left[ \mu'(z) \frac{r}{z_x} + \frac{1}{6} \partial_x \left( \frac{r^3}{z_x^3} \right) \right], \\ r_t &= \partial_x \left[ \mu'(z) + \partial_x \left( \frac{r^2}{2z_x^2} \right) \right], & \partial_t(rz) &= \partial_x \left[ z\mu'(z) - \mu(z) + \frac{z}{2} \partial_x \left( \frac{r^2}{z_x^2} \right) \right]. \end{aligned}$$

## 6. Recursion Operator. Integrability of the GZ hierarchy

Applying the reciprocal transformation (8) simultaneously to both Lagrangian representations (22) and (24), one obtains the variation principles in *another* independent variables

$$S_1 = \int [\frac{1}{2}x_\tau^2 + \mu''(z)x_z]dzd\tau \quad \text{and} \quad S_2 = \int [\frac{1}{2}\tilde{\Phi}_\tau^2 - \mu(z)\tilde{\Phi}_{zz}]dzd\tau,$$

where  $u = x_\tau$ ,  $\rho^{-1} = x_z$ ,  $x = \tilde{\Phi}_z$ , (see the first conservation law associated with the second Hamiltonian structure (25):  $d\Phi = zdx + \frac{r}{\rho}dt$  or  $d\tilde{\Phi} = xdz - \frac{r}{\rho}d\tau$ , where  $xz = \Phi + \tilde{\Phi}$ ). Then the Euler-Lagrange equation is  $x_{\tau\tau} = -\mu'''(z)$ . This equation can be easily integrated (see (10)). Corresponding Poisson brackets (see (19) and (25))

$$\begin{aligned} \{x(z), u(z')\}_1 &= -\{u(z), x(z')\}_1 = \delta(z - z'), \\ \{p(z), \tilde{\Phi}(z')\}_2 &= -\{\tilde{\Phi}(z), p(z')\}_2 = \delta(z - z'), \end{aligned}$$

where  $u = \tilde{\Phi}_{z\tau}$  and  $d\tilde{\Phi} = xdz - pd\tau$  (i.e.  $p = r/\rho \equiv \Phi_t$ ), create the recursion operator

$$\hat{R} = - \begin{pmatrix} \partial_z^2 & \\ & \partial_z^2 \end{pmatrix}$$

where

$$\{x(z), u(z')\}_2 = -\{u(z), x(z')\}_2 = -\delta''(z - z').$$

Thus, the Gurevich-Zybin system in these independent variables has an *infinite set of local Hamiltonian structures, conservation laws and commuting flows*. For instance, all such Hamiltonians are

$$\tilde{H}_k = (-1)^k \int [\frac{1}{2}p^{(k)^2} + \mu^{(k+2)}(z)\tilde{\Phi}^{(k)}]dz, \quad k = 0, \pm 1, \pm 2, \dots$$



Corresponding commuting flows

$$\tilde{\Phi}_{\tau^k \tau^k} = -\mu^{(k+2)}(z)$$

can be easily integrated (see (10)). However, in the independent variables  $(x, t^k)$  they can be written in the form (cf. (7))

$$\partial_{t^k} u^k + u^k \partial_x u^k + \mu^{(2k+3)}(z) = 0, \quad \partial_{t^k} z + u^k \partial_x z = 0, \quad (26)$$

where

$$u^0 \equiv u, \quad u^{k+1} = \frac{1}{\rho} \partial_x u^k, \quad u^{-k-1} = \partial_x^{-1}(\rho u^{-k}), \quad k = 0, 1, 2, \dots \quad (27)$$

Thus, *all commuting flows* to the Gurevich-Zybin system created by above bi-Hamiltonian structure are *Monge-Ampere equations* (cf. (23))

$$\Phi_{xx} \Phi_{t^k t^k} - \Phi_{xt^k}^2 = \mu^{(2k+2)}(\Phi_x) \Phi_{xx},$$

where

$$u^k = -\Phi_{t^{k+1}}, \quad \Phi_{xt^k} = \Phi_{xx} \Phi_{t^{k+1}}, \quad k = 0, \pm 1, \pm 2, \dots$$

All local Lagrangians are

$$\begin{aligned} S_{2,k} &= \int \left[ \frac{1}{2} \Phi_{xx} \Phi_{t^k}^2 - \mu^{(2k)}(\Phi_x) \right] dx dt^k, \\ S_{1,k} &= \int \left[ \frac{\Phi_{xt^k}^2}{2\Phi_{xx}} + \mu^{(2k+2)}(\Phi_x) \right] dx dt^k, \\ S_{0,k} &= \int \left[ \frac{1}{2\Phi_{xx}} \left[ \left( \frac{\Phi_{xt^k}}{\Phi_{xx}} \right)_x \right]^2 - \mu^{(2k+4)}(\Phi_x) \right] dx dt^k, \\ S_{-1,k} &= \int \left[ \frac{1}{2\Phi_{xx}} \left[ \left( \frac{1}{\Phi_{xx}} \left[ \left( \frac{\Phi_{xt^k}}{\Phi_{xx}} \right)_x \right] \right)_x \right]^2 + \mu^{(2k+6)}(\Phi_x) \right] dx dt^k, \dots \end{aligned}$$

**Remark 2:** All commuting flows have infinitely many different *local* representations via different pairs of field variables  $(z, u^k)$ , see (27). For instance (cf. (26))

$$\begin{aligned} z_{t^k} + \partial_x u^{k-1} &= 0, \quad \partial_{t^k} u^{k-1} + \frac{(u_x^{k-1})^2}{z_x} + \mu^{(2k+2)}(z) = 0, \\ z_{t^k} + \partial_x \left( \frac{u_x^{k-2}}{z_x} \right) &= 0, \quad \partial_{t^k} u^{k-2} + \partial_x \left[ \frac{(u_x^{k-2})^2}{2z_x^2} \right] + \mu^{(2k+1)}(z) = 0. \end{aligned}$$

The theory of integrable systems with a multi-Lagrangian structure is presented in [23]. Usually, every local Lagrangian creates a nonlocal Hamiltonian structure. Such explicit formulas of nonlocal Hamiltonian structures, nonlocal commuting flows, nonlocal conservation laws as well as nonlocal Lagrangians can be found iteratively from already given above formulas.

## 7. Another bi-Hamiltonian structure

Now in this and two next sections we identify  $v \equiv z$ , i.e. we concentrate attention on case  $\mu'''(z) = z$ , see (4)). In two previous sections we discuss bi-Hamiltonian structure of the Gurevich-Zybin system. Here we preserve the first Hamiltonian structure (see (18), (21)), but change the second one! In above section we proved that the Gurevich-Zybin system has infinitely many local Hamiltonian structures and Lagrangian representations (a

general theory is presented in [23], see also [22]). However, this new second Hamiltonian structure (see below) is **not** from this set!

The Gurevich-Zybin system (2) is an Euler-Lagrange equation of corresponding variational principle (see (22), when  $\mu''(z) = z^2/2$ )

$$S = \frac{1}{2} \int \left[ \frac{z_t^2}{z_x} + z^2 \right] dx dt.$$

However, the *astonished* fact is that the *Gurevich-Zybin system* (2) has **another** *Hamiltonian structure connected with the same Lagrangian density*. Namely (see for details [22], especially formulas (43), (52-54) therein), the Lagrangian (cf. with  $S$ )

$$\tilde{S} = \frac{1}{2} \int \left[ \frac{p_x}{z_x} (2z_t - p_x) + z^2 \right] dx dt$$

determines the same Euler-Lagrange equations (2) but with another Hamiltonian structure

$$u_t = -\partial_x^{-1} \frac{\delta H_1}{\delta u} + u_x \partial_x^{-1} \frac{\delta H_1}{\delta z}, \quad z_t = \partial_x^{-1} \left( u_x \frac{\delta H_1}{\delta u} + z_x \frac{\delta H_1}{\delta z} \right) + z_x \partial_x^{-1} \frac{\delta H_1}{\delta z},$$

where  $u = -p_x/z_x$  (i.e.  $p = \Phi_t$ ).

**Remark 3:** This bi-Hamiltonian structure at first was discovered by Yavuz Nutku [21] and later it was independently found in [6] (see formula (9) therein) exactly as it was done in [22]. However, here we repeat and emphasize the **main observation** of this section is that **both** *Hamiltonian structures have the same Lagrangian density!* This is the first such example in the theory of integrable systems.

**Canonical representation for both *Hamiltonian* structures and recursion operator.** The Poisson bracket

$$\begin{aligned} \{u(x), u(x')\}_1 &= 0, & \{\rho(x), u(x')\}_1 &= \delta'(x - x'), \\ \{u(x), \rho(x')\}_1 &= \delta'(x - x'), & \{\rho(x), \rho(x')\}_1 &= 0 \end{aligned}$$

of the first Hamiltonian structure is given in its canonical form (more details see in the review [10]). However, the Poisson bracket

$$\begin{aligned} \{u(x), u(x')\}_2 &= -\partial^{-1} \delta(x - x'), & \{u(x), \rho(x')\}_2 &= -u_x \delta(x - x'), \\ \{\rho(x), u(x')\}_2 &= u_x \delta(x - x'), & \{\rho(x), \rho(x')\}_2 &= -(\rho \partial_x + \partial_x \rho) \delta(x - x') \end{aligned}$$

of the second Hamiltonian structure can be reduced by the Darboux theorem to the canonical form

$$\begin{aligned} \{w(x), w(x')\}_2 &= \delta'(x - x'), & \{\eta(x), w(x')\}_2 &= 0, \\ \{w(x), \eta(x')\}_2 &= 0, & \{\eta(x), \eta(x')\}_2 &= \delta'(x - x') \end{aligned}$$

by the Miura type transformation (see the first equation in (16))

$$w = u_x, \quad \rho = -\frac{1}{2}(w^2 + \eta^2).$$

Then the Gurevich-Zybin system written in a *modified* form (see formula (26) in [26], other details in the last section 5 and references [6], [7] therein)

$$\eta_t + \partial_x(u\eta) = 0, \quad u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = \frac{1}{2}\eta^2 \quad (28)$$

can be recognized as the two-component generalization of the Hunter-Saxton equation (cf. [15], [8], [25]).

**Remark 4:** In fact, the Casimir density  $\eta$  of the second Hamiltonian structure was found in [6] (see formula (18) therein). However, the Gurevich-Zybin system was not presented in form (28) there. Moreover, we emphasize the **main result** of this paper is that *the Gurevich-Zybin system belongs to the well-known class of integrable systems*. In this section we prove that the Gurevich-Zybin system is a member of an integrable hierarchy embedded into 2x2 spectral transform.

Since the first Poisson bracket in new field variables has a form

$$\begin{aligned}\{w(x), \eta(x')\}_1 &= -\left[\frac{1}{\eta}\delta(x-x')\right]'', & \{\eta(x), w(x')\}_1 &= \frac{1}{\eta}\delta''(x-x'), \\ \{w(x), w(x')\}_1 &= 0, & \{\eta(x), \eta(x')\}_1 &= -\left[\frac{w_x}{\eta^2}\partial_x + \partial_x\frac{w_x}{\eta^2}\right]\delta(x-x'),\end{aligned}$$

then the modified Gurevich-Zybin system (28) as a member of integrable hierarchy with all other commuting flows together can be written in bi-Hamiltonian form

$$\begin{aligned}w_{t^k} &= \partial_x \frac{\delta H_{k+1}}{\delta w} = -\partial_x^2 \left[ \frac{1}{\eta} \cdot \frac{\delta H_k}{\delta \eta} \right], \\ \eta_{t^k} &= \partial_x \frac{\delta H_{k+1}}{\delta \eta} = \frac{1}{\eta} \partial_x^2 \frac{\delta H_k}{\delta w} - \left[ 2 \frac{w_x}{\eta^2} \partial_x + \left( \frac{w_x}{\eta^2} \right)_x \right] \frac{\delta H_k}{\delta \eta},\end{aligned}$$

An eigenvalue problem for the recursion operator as a ratio of both Hamiltonian structures

$$\begin{bmatrix} 0 & -\partial_x^2 \frac{1}{\eta} \\ \frac{1}{\eta} \partial_x^2 & -\left( \frac{w_x}{\eta^2} \partial_x + \partial_x \frac{w_x}{\eta^2} \right) \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 2\lambda \partial_x \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

can be written as one equation

$$\varphi_{xxx} + 4(\lambda^2 \eta^2 + \lambda \sigma) \varphi_x + 2(\lambda^2 \eta^2 + \lambda \sigma)_x \varphi = 0,$$

where  $\varphi_1 = \varphi_x$ ,  $\varphi_2 = -2\lambda\eta\varphi$  and  $\sigma = w_x$ . However, above equation can be reduced to

$$\psi_{xx} + (\lambda^2 \eta^2 + \lambda \sigma) \psi = 0,$$

where  $\varphi = \psi\psi^+$  is a *squared eigenfunction* and  $\psi, \psi^+$  are linear conjugate solutions with different asymptotics at infinity  $\lambda \rightarrow \infty$ . This linear spectral problem (more precisely, just “ $x$ ”-dynamics) is well known in the theory of integrable systems: corresponding systems are members (commuting flows) of the two-component Harry Dym hierarchy (see for instance [2]). All such members of this hierarchy can be determined by the spectral transform

$$\psi_{xx} = -(\lambda^2 \eta^2 + \lambda \sigma) \psi, \quad \psi_t = b\psi_x - \frac{1}{2}b_x \psi, \quad (29)$$

where  $b(\zeta, \eta, \lambda)$  is a *polynomial* function with respect to the spectral parameter  $\lambda$  for *positive* members. The compatibility condition  $(\psi_{xx})_t = (\psi_t)_{xx}$  yields relationship

$$(\lambda^2 \eta^2 + \lambda \sigma)_t = \left[ \frac{1}{2} \partial_x^3 + 2(\lambda^2 \eta^2 + \lambda \sigma) \partial_x + (\lambda^2 \eta^2 + \lambda \sigma)_x \right] b,$$

where the two-component Harry Dym system (see [2])

$$\eta_{t_1} = \left( \frac{\sigma}{\eta^2} \right)_x, \quad \sigma_{t_1} = \left( \frac{1}{\eta} \right)_{xxx}$$

can be obtained if  $b = 2\lambda/\eta$ . Thus, *twice potential* two-component Harry Dym system

$$\eta_{t_1} = \left(\frac{u_{xx}}{\eta^2}\right)_x, \quad u_{t_1} = \left(\frac{1}{\eta}\right)_x \quad (30)$$

is the *first* member of *positive* part of above hierarchy and the *first* member of its *negative* part is the *modified* Gurevich-Zybin system (28)

$$\eta_{t_{-1}} + \partial_x(u\eta) = 0, \quad u_{xt_{-1}} + uu_{xx} + \frac{1}{2}u_x^2 = \frac{1}{2}\eta^2 \quad (31)$$

determined by the choice  $b = (2\lambda)^{-1} - u$  (it means that we **must** identify  $t \equiv t_{-1}$  for the Gurevich-Zybin system (2)).

**Remark 5:** The reciprocal transformation (see (8))

$$d\tau_1 = dt_1, \quad d\tau_{-1} = dt_{-1}, \quad dz = \rho dx - \rho u dt_{-1} - \left(\frac{u_x}{\eta}\right)_x dt_1$$

simultaneously linearizes the Gurevich-Zybin system (see (9) and (10)) and *preserves* two-component Harry Dym system:

$$\rho_{t_1} = -\left(\frac{w}{\eta}\right)_{xx}, \quad w_{t_1} = \left(\frac{1}{\eta}\right)_{xx}, \quad \eta_{t_1} = \left(\frac{w_x}{\eta^2}\right)_x \rightarrow \bar{\rho}_{\tau_1} = -\left(\frac{\bar{w}}{\bar{\eta}}\right)_{zz}, \quad \bar{w}_{\tau_1} = \left(\frac{1}{\bar{\eta}}\right)_{zz}, \quad \bar{\eta}_{\tau_1} = \left(\frac{\bar{w}_z}{\bar{\eta}^2}\right)_z,$$

where

$$\bar{\rho} = \frac{1}{\rho}, \quad \bar{w} = -\frac{w}{\rho}, \quad \bar{\eta} = \frac{\eta}{\rho}.$$

Such reciprocal *auto-transformation* is the first example in the theory of integrable systems.

**Remark 6:** Twice potential two-component Harry Dym system (30) written in field variables  $(\rho, u)$  was also found in [6] (see formula (21) therein), but was not *recognized*.

## 8. Reciprocal and Miura type transformations

Application of the reciprocal transformation (in fact, it was given in [2], formulas (32-34) therein; cf. (13))

$$dy_1 = dt_1, \quad dy_{-1} = dt_{-1}, \quad d\zeta = \eta dx - \eta u dt_{-1} + \frac{u_{xx}}{\eta^2} dt_1 \quad (32)$$

to the spectral transform (29) yields another well-known spectral transform (see for instance, [2], [27]), where “ $\zeta$ ”-dynamics is

$$\tilde{\psi}_{\zeta\zeta} + [\lambda^2 - \tilde{u}\lambda - \tilde{v} + \frac{\tilde{u}^2}{4}]\tilde{\psi} = 0, \quad (33)$$

“ $y$ ”-dynamics is

$$\tilde{\psi}_{y_1} = (2\lambda + \tilde{u})\tilde{\psi}_\zeta - \frac{1}{2}\tilde{u}_\zeta\tilde{\psi}, \quad \tilde{\psi}_{y_{-1}} = \frac{1}{4\lambda}(2\eta\tilde{\psi}_\zeta - \eta_\zeta\tilde{\psi}) \quad (34)$$

and

$$\tilde{\psi} = \eta^{1/2}\psi, \quad -\tilde{u} = u_{\zeta\zeta} + \frac{\eta_\zeta}{\eta}u_\zeta, \quad -\tilde{v} + \frac{\tilde{u}^2}{4} = \frac{\eta_\zeta^2}{4\eta^2} - \frac{\eta_{\zeta\zeta}}{2\eta}. \quad (35)$$

The compatibility conditions  $(\tilde{\psi}_{\zeta\zeta})_{y_1} = (\tilde{\psi}_{y_1})_{\zeta\zeta}$  and  $(\tilde{\psi}_{\zeta\zeta})_{y_{-1}} = (\tilde{\psi}_{y_{-1}})_{\zeta\zeta}$  yield the first *positive* member

$$\tilde{u}_{y_1} = 2\partial_\zeta\left[\frac{\tilde{u}^2}{2} + \tilde{v}\right], \quad \tilde{v}_{y_1} = 2\partial_\zeta\left[\tilde{u}\tilde{v} - \frac{1}{4}\tilde{u}_{\zeta\zeta}\right], \quad (36)$$

and the first *negative* member

$$\tilde{u}_{y-1} = -\eta_\zeta, \quad \tilde{v}_{y-1} = \frac{1}{2}\partial_\zeta(\tilde{u}\eta), \quad -\frac{1}{2}\eta_{\zeta\zeta\zeta} + (2\tilde{v} - \frac{\tilde{u}^2}{2})\eta_\zeta + (\tilde{v}_\zeta - \frac{1}{2}\tilde{u}\tilde{u}_\zeta)\eta = 0$$

which also can be obtain by the limit  $\tilde{\lambda} \rightarrow 0$  from the generating function of commuting flows

$$\tilde{u}_y = -\tilde{\eta}_\zeta, \quad \tilde{v}_y = \partial_\zeta[(\frac{1}{2}\tilde{u} - \tilde{\lambda})\tilde{\eta}],$$

$$\tilde{\eta}_{\zeta\zeta\zeta} + 4(\tilde{\lambda}^2 - \tilde{u}\tilde{\lambda} - \tilde{v} + \frac{\tilde{u}^2}{4})\tilde{\eta}_\zeta + 2(-\tilde{\lambda}\tilde{u}_\zeta - \tilde{v}_\zeta + \frac{1}{2}\tilde{u}\tilde{u}_\zeta)\tilde{\eta} = 0, \quad (37)$$

where  $\tilde{\eta} = \tilde{\varphi}\tilde{\varphi}^+$  is a *squared eigenfunction* and  $\tilde{\varphi}$ ,  $\tilde{\varphi}^+$  are linear conjugate solutions of the spectral transform (33) with different asymptotics at the infinity  $\tilde{\lambda} \rightarrow \infty$  (see details for instance in [28]).

At the same time, as it was proved in [26] (see also above section 3), the modified Gurevich-Zybin system is *linearized* by above reciprocal transformation (32). Simultaneously, the *twice potential* two-component Harry Dym system (30) transforms into *twice degenerate twice modified* Kaup-Boussinesq system

$$u_{y_1} = -u_\zeta u_{\zeta\zeta} - \frac{\eta_\zeta}{\eta}(1 + u_\zeta^2), \quad \eta_{y_1} = \eta u_{\zeta\zeta\zeta} + (\eta_{\zeta\zeta} - 2\frac{\eta_\zeta^2}{\eta})u_\zeta. \quad (38)$$

It is well-known that (36) is the Kaup-Boussinesq system (see for instance [18]). Several modified Kaup-Boussinesq systems were presented in [27]. The *modified* Kaup-Boussinesq system is

$$\tilde{u}_{y_1} = \partial_\zeta[\frac{3}{2}\tilde{u}^2 + 2u_1^2 + 2u_{1,\zeta}], \quad u_{1,y_1} = \partial_\zeta[u_1\tilde{u} - \frac{1}{2}\tilde{u}_\zeta],$$

where the *first* Miura transformation is

$$\tilde{v} = \frac{1}{4}\tilde{u}^2 + u_1^2 + u_{1,\zeta}. \quad (39)$$

The *twice modified* Kaup-Boussinesq system is

$$u_{1,y_1} = \partial_\zeta[(2u_1^2 - u_{1,\zeta})u_2 - \frac{1}{2}u_{2,\zeta\zeta}], \quad u_{2,y_1} = \partial_\zeta[2u_1(1 + u_2^2) + u_2u_{2,\zeta}],$$

where the *second* Miura transformation is

$$\tilde{u} = 2u_1u_2 + u_{2,\zeta}. \quad (40)$$

It was proved in [27] that the Kaup-Boussinesq system has third and fourth Miura transformations (see also [5]). Their *double parametric degeneration* to *purely potential* form (cf. (39), (40) with second and third equations from (35))

$$u_1 = \frac{1}{2}\partial_\zeta \ln \eta, \quad u_2 = -u_\zeta \quad (41)$$

transforms the twice modified Kaup-Boussinesq system in form (38).

Thus, the **main result** of this section is an establishment of the link of transformations (reciprocal and Miura type) between the Gurevich-Zybin and the Kaup-Boussinesq hierarchies.

## 9. Second Harmonic Generation

The generation of the second harmonic wave from the red light of a ruby laser in a crystal of quartz in fact was the starting point of nonlinear optics. In one-dimensional case, for short pulses, when the group-velocity mismatch between both frequency components becomes important, then the process of second harmonic generation (SHG) is described by the complex equations (see for instance [19], [20]), which in real form are (see [20], formula (6) therein)

$$\tilde{u}_{y-1} = -\eta_\zeta = -2\eta u_1, \quad 2u_{1,y-1} = -\frac{1}{\eta} + \eta\tilde{u}. \quad (42)$$

The corresponding spectral problem (see again [20], formula (9) therein)

$$\tilde{\psi}_{\zeta\zeta} + [\lambda^2 - \tilde{u}\lambda - u_1^2 - u_{1,\zeta}]\tilde{\psi} = 0, \quad \tilde{\psi}_{y-1} = \frac{1}{2\lambda}\eta\tilde{\psi}_\zeta - \frac{1}{2\lambda}\eta u_1\tilde{\psi}$$

is a *special* reduction of the spectral transform (33), (34). In general case (37) can be integrated once

$$\tilde{\eta}\tilde{\eta}_{\zeta\zeta} - \frac{1}{2}\tilde{\eta}_\zeta^2 + 2(\tilde{\lambda}^2 - \tilde{u}\tilde{\lambda} - \tilde{v} + \frac{\tilde{u}^2}{4})\tilde{\eta}^2 + S(\tilde{\lambda}) = 0,$$

where  $S(\tilde{\lambda})$  is a polynomial function for multi-periodic solutions of the Kaup-Boussinesq hierarchy (see for instance [1]). Thus, the first *negative* member of this hierarchy has the constraint

$$\eta\eta_{\zeta\zeta} - \frac{1}{2}\eta_\zeta^2 + 2(-\tilde{v} + \frac{\tilde{u}^2}{4})\eta^2 + S_{-1} = 0,$$

where  $S_{-1} \neq 0$  is a some constant. However,  $S_{-1} = 0$  in the case of the SHG system! It is easy to prove by direct substitution (39) and  $\eta_\zeta = 2\eta u_1$  from (42) (see also the first equation in (41)) in above equation. Thus, the SHG system (42) is the *degenerate first negative* member of the *modified* Kaup-Boussinesq hierarchy (see (39) and above).

The SHG system

$$(\ln \Xi_{y-1})_{\zeta y-1} = \frac{1}{\Xi_{y-1}} - \Xi_\zeta \Xi_{y-1} \quad (43)$$

can be interpreted as the two-component generalization of the Sinh-Gordon equation, where  $\Xi_\zeta = \tilde{u}$  and  $\Xi_{y-1} = -\eta$ . The SHG system has *three* different linearizable degenerations, as well as the Sinh-Gordon equation has a *parametric* degeneration to the famous Liouville equation which is linearizable. First two degenerate limits are known (see [3] and [17]). These are the Liouville equation and the modified Liouville equation. The *third* such case can be obtained by (see above) differential substitutions (40) and (41). This is the modified Gurevich-Zybin system (31) re-calculated by the reciprocal transformation (32) (see formula (27) in [26] and other details in the last section 5)

$$u_\zeta = (\frac{1}{\eta})_{y-1}, \quad \eta_{y-1 y-1} - \frac{3\eta_{y-1}^2}{2\eta} + \frac{1}{2}\eta^3 = 0.$$

Thus, a solution of the *reduced* SHG system

$$(u_{\zeta\zeta} + \frac{\eta_\zeta}{\eta}u_\zeta)_{y-1} = \eta_\zeta, \quad (\ln \eta)_{\zeta y-1} = -\frac{1}{\eta} - (\eta u_\zeta)_\zeta$$

in an implicit form is given by (17)

$$\begin{aligned}\eta &= \left[ \frac{1}{\eta_0(\zeta)} + \frac{\eta_0(\zeta)}{4} (y_{-1} - y_0(\zeta))^2 \right]^{-1}, \\ u &= \frac{y_{-1}}{2} \int \eta_0(\zeta) d\zeta - \frac{1}{2} \int \eta_0(\zeta) y_0(\zeta) d\zeta, \\ x &= \int \left[ \frac{1}{\eta_0(\zeta)} + \eta_0(\zeta) y_0^2(\zeta) \right] d\zeta - \frac{y}{2} \int \eta_0(\zeta) y_0(\zeta) d\zeta + \frac{y^2}{4} \int \eta_0(\zeta) d\zeta.\end{aligned}$$

Thus, a *new* solution of the SHG system (43) can be found in quadratures

$$d\Xi = -(u_{\zeta\zeta} + \frac{\eta_{\zeta}}{\eta} u_{\zeta}) d\zeta - \eta dy_{-1}.$$

**Final remark:** Since the Kaup-Boussinesq system and the nonlinear Schroedinger equation are related by invertible transformations (see for instance [1]), then their first negative flows are related too. Since the first negative flow to the nonlinear Schroedinger equation is another famous Maxwell-Bloch system (in particular case the “self-induced transparency” coincides with the Maxwell-Bloch system), then the SHG system connected to Maxwell-Bloch system by the same transformations. Thus, the Gurevich-Zybin system is connected with the Maxwell-Bloch system. Similarly, a new solution of the Maxwell-Bloch system can be found by the same way. Since nonlinear Schroedinger equation relates to Heisenberg magnet by Miura type transformations, then particular case of the “Ramann scattering” is connecting with the Maxwell-Bloch system too. Thus, a new solution for the Ramann scattering can be constructed as in previous case.

## 10. Open Problems

The numerical simulation of nonlinear dynamics described by the Gurevich-Zybin system yields the hypothesis that a behaviour in multimode form demonstrating the transition from the hydrodynamic to the equilibrium kinetic state has some regular features (see for details [14]) possibly generated by *integrable* properties of corresponding  $N$ -component Gurevich-Zybin system (see (3))

$$\rho_t^k + \partial_x(\rho^k u^k) = 0, \quad u_t^k + u^k u_x^k + \Phi_x = 0, \quad \Phi_{xx} = \sum_{m=1}^N \rho^m.$$

This problem (integrability in any sense: linearization, inverse scattering transform, bi-Hamiltonian formulation, etc) is open. For instance, this system written in field variables  $u^k$  and  $v^k$  ( $\rho^k \equiv v_x^k$ ) has ultra-local Hamiltonian structure

$$v_t^k = \frac{\delta H_2}{\delta u^k}, \quad u_t^k = -\frac{\delta H_2}{\delta v^k},$$

where the Hamiltonian is

$$H_2 = \frac{1}{2} \int \left[ -\sum_{m=1}^N (u^m)^2 v_x^m + \left( \sum_{m=1}^N v^m \right)^2 \right] dx.$$

It was proved here that the Gurevich-Zybin system has infinitely many Hamiltonian structures. Existence of the second Hamiltonian structure is enough for an integrability.

$$A_k = \sum_{m=1}^N (u^m)^k \rho^m \quad (44)$$

then  $N$ -component Gurevich-Zybin system can be written as the nonlocal chain

$$\partial_t A_k + \partial_x A_{k+1} + k A_{k-1} \partial_x^{-1} A_0 = 0, \quad k = 0, 1, 2, \dots, \quad (45)$$

which looks very similar as famous *integrable* Benney moment chain (see [4])

$$\partial_t A_k + \partial_x A_{k+1} + k A_{k-1} \partial_x A_0 = 0, \quad k = 0, 1, 2, \dots$$

The Benney moment chain has infinitely many  $N$ -component reductions (see [13]) parameterized by  $N$  functions of a single variable ( $N$  is an arbitrary natural integer), where the simplest reduction is (44). The nonlocal chain (45) has at least one simple reduction

$$A_k = \rho u^k.$$

The existence of any other such reductions could be a *symptom* of an integrability. The integrability of  $N$ -component Gurevich-Zybin system and a description of other reductions of the nonlocal chain (45) will be under consideration anywhere else.

## 11. Conclusion

The Gurevich-Zybin system is an example of integrable systems possessing properties of two different classes:  $C$ - and  $S$ -integrable. This system is linearizable (see [14]) and has a general solution. Thus, the Gurevich-Zybin system is from a  $C$ -integrable class. However, this system has an infinite set of Hamiltonian structures and commuting flows. Thus, the Gurevich-Zybin system is from a  $S$ -integrable class too. Moreover, this system has an infinite set of *local* Hamiltonian structures, that is unusual in the theory of  $S$ -integrable systems. Moreover, all commuting flows of the Gurevich-Zybin system written in a form of a Monge-Ampere equation are the same Monge-Ampere equation again. The difference between them is just some derivative of function  $\mu(z)$ , which can be eliminated by a point transformation  $v = \mu'''(z)$  (see (7)). Thus, this is a beautiful example from mathematical point of view having physical application (see [14]). We repeat again: *every commuting flow* can be written in the *unique form* (23)

$$\Phi_{xx} \Phi_{t^k t^k} - \Phi_{xt^k}^2 = K(\Phi_x) \Phi_{xx},$$

where  $K(z)$  is **any** apriori given function, but all commuting flows will be written via different functions  $\Phi_{(k)}$ , because the point transformation like  $v = \mu'''(z)$  becomes nonlocal

$$K(\Phi_{(k)x}) = \mu^{(2k+2)}(\Phi_x).$$

The exceptional case is when some derivative  $\mu^{(n)}(z)$  is a constant: then “half” of commuting flows are trivial (see [22], [11])

$$\Phi_{xx} \Phi_{t^k t^k} - \Phi_{xt^k}^2 = 0,$$

when  $n$  is even, then  $k \geq n/2$ , when  $n$  is odd, then  $k \geq (n-1)/2$ .

At past 10 years a couple of such examples of integrable systems (mixed properties of  $C$ - and  $S$ -integrability) was found in [22] and [12]. However, an explanation of such phenomenon is not exist to this moment. One possible explanation is that such systems are in *intersection* of  $C$ - and  $S$ -integrability. Thus, they *accumulate* properties of these two different classes. Moreover, we proved that the Gurevich-Zybin system is a



*degenerate* member of the two-component Harry Dym hierarchy. A *degeneracy* becomes when equations can possess a *parametric* freedom. When some of parameters are fixed (to zero, for instance), then such equations become to *linearizable*. The simplest example is the famous Liouville equation

$$w_{xt} = e^w.$$

This equation is in *intersection* of two different integrable hierarchies. One of them is another famous Bonnet equation (well known in physics as the Sinh-Gordon equation) first introduced in a differential geometry of surfaces of a constant curvature

$$w_{xt} = c_1 e^w + c_2 e^{-w},$$

which is a member of the potential *modified* KdV hierarchy (spectral transform 2x2)

$$w_\tau = w_{xxx} - \frac{1}{2}w_x^3.$$

Another one is the Tzitzeica equation, well known in an affine differential geometry

$$w_{xt} = c_1 e^w + c_2 e^{-2w},$$

which is a member of the potential *modified* Sawada-Kotera hierarchy (spectral transform 3x3)

$$w_\tau = w_{xxxxx} + 5(w_{xx}w_{xxx} - w_x^2 w_{xxx} - w_x w_{xx}^2) + w_x^5.$$

Thus, if  $c_2 = 0$ , then the Liouville equation is still *a member of two different integrable hierarchies simultaneously*. This is a good *symptom* that such equations should be linearizable. Since above mentioned linearizable reduction of the SHG system is determined by 2x2 spectral transform, but the SHG system is a some reduction of another important three-wave interaction problem (see for instance [20]), then we can assume such systems like the Gurevich-Zybin system are linearizable if they are in intersection of at least two different integrable hierarchies (e.g. the Liouville equation is a member of two different hierarchies: of the KdV equation and of the Sawada-Kotera equation). A bi-Hamiltonian structures presented here has origin in 2x2 spectral transform. It will be interesting to find another Hamiltonian structures coming from 3x3 spectral problem.

Finally, we would like to emphasize that this paper was devoted to **recognition** of a relationship between couple of remarkable systems having applications in astrophysics, nonlinear optics and geometry.

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